Hicksian Demand and Expenditure Function Duality, Slutsky Equation

Econ 3030 Fall 2025

Lecture 6

Outline

1. Applications of Envelope Theorem
2. Hicksian Demand
3. Duality
4. Connections between Walrasian and Hicksian demand functions.
5. Slutsky Decomposition: Income and Substitution Effects

Comparative Statics With Constraints

We solve $\displaystyle\max_{F(\mathbf{x};\mathbf{q})=\mathbf{0}_k} \phi(\mathbf{x};\mathbf{q})$ using the Lagrangian: $L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q}) = \phi(\mathbf{x};\mathbf{q}) - \boldsymbol{\lambda}^\top F(\mathbf{x};\mathbf{q})$

- The FOC is $\boxed{D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q}) = \mathbf{0}}$ (we apply IFT to this equation)

Comparative Statics from last class

Fix some $\overline{\mathbf{q}}$ and let $\boldsymbol{\lambda}^*(\overline{\mathbf{q}})$ and $\mathbf{x}^*(\overline{\mathbf{q}})$ be solutions to the FOC (and thus optimal choices): by IFT, in a neighborhood of $\overline{\mathbf{q}}$:

$$D_{\mathbf{q}}(\lambda^*(\overline{\mathbf{q}}), x^*(\overline{\mathbf{q}})) = -\frac{D_{\mathbf{q}}\left[D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]}{D_{(\boldsymbol{\lambda},\mathbf{x})}\left[D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]}$$

- The rest is just algebra: calculate
  - the cross derivative $D_{\mathbf{q}}\left[D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]$ and
  - the second derivative $D_{(\boldsymbol{\lambda},\mathbf{x})}\left[D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]$
    - where $D_{(\boldsymbol{\lambda},\mathbf{x})} L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q}) = \begin{pmatrix} -F(\mathbf{x};\mathbf{q}) \\ D_{\mathbf{x}}\phi(\mathbf{x};\mathbf{q}) - \boldsymbol{\lambda}^\top D_{\mathbf{x}} F(\mathbf{x};\mathbf{q}) \end{pmatrix}$ which is a $k+n$ vector

Envelope Theorem (With Constraints)

Envelope Theorem from last class

The Envelope Theorem is:

$$D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = D_q\phi(\mathbf{x},\mathbf{q})\big|_{\mathbf{q}=\overline{\mathbf{q}},\ \mathbf{x}=x^*(\overline{\mathbf{q}})} - \left(\lambda^*(\overline{\mathbf{q}})\right)^\top D_{\mathbf{q}}F(\mathbf{x},\mathbf{q})\big|_{\mathbf{q}=\overline{\mathbf{q}},\ x=x(\overline{\mathbf{q}})}$$

- The direct effect of the parameter $\mathbf{q}$ is on both the value of ϕ evaluated at the maximizer $x^*(\overline{\mathbf{q}})$ but also on the constraints.

Summary From Last Class

Summary from last class

Fix some $\overline{\mathbf{q}}$ and let $\boldsymbol{\lambda}^*(\overline{\mathbf{q}})$ and $\mathbf{x}^*(\overline{\mathbf{q}})$ be solutions to the FOC (and thus optimal choices): By IFT, in a neighborhood of $\overline{\mathbf{q}}$:

$$D_{\mathbf{q}}(\lambda^*(\overline{\mathbf{q}}), x^*(\overline{\mathbf{q}})) = -\frac{D_{\mathbf{q}}\left[D_{(\boldsymbol{\lambda},\mathbf{x})}L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]}{D_{(\boldsymbol{\lambda},\mathbf{x})}\left[D_{(\boldsymbol{\lambda},\mathbf{x})}L(\boldsymbol{\lambda},\mathbf{x};\mathbf{q})\right]}$$

The Envelope Theorem is:

$$D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = D_q\phi(\mathbf{x},\mathbf{q})\big|_{\mathbf{q}=\overline{\mathbf{q}},\ x=x^*(\overline{\mathbf{q}})} - (\lambda^*(\overline{\mathbf{q}}))^\top D_{\mathbf{q}}F(\mathbf{x},\mathbf{q})\big|_{q=\overline{\mathbf{q}},\ x=x(\overline{\mathbf{q}})}$$

- In a utility maximization problem: utility does not depend on exogenous variables so the differential effect of price and wage changes is via the budget constraint
  - $\phi(\cdot;\cdot)$ is the utility function $u(\cdot)$ and therefore $\phi(x^*(\cdot);\cdot)$ is $u(x^*(\cdot)) = v(\mathbf{p}, w))$
  - F is the budget constraint $\mathbf{p}\cdot\mathbf{x} - w$;
- Figure out how the expressions above would work if (some) prices and income change.

Exercise

Compute $\frac{\partial x^*(p,w)}{\partial p_k}$ with $k = 1, 2$ for the Cobb–Douglas utility function on $\mathbb{R}^2_+$.

Application: Roy's Identity

Walrasian demand is $\boxed{x^*(\mathbf{q}) = \arg\max u(\mathbf{x}) \text{ subject to } \mathbf{p}\cdot\mathbf{x} - w = 0}$

- Apply the previous results with
  $\phi(\mathbf{x},\mathbf{q}) = u(\mathbf{x})$, $\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = u(x^*(\mathbf{p},w)) = v(\mathbf{p},w)$, and $F(\mathbf{x},\mathbf{q}) = \mathbf{p}\cdot\mathbf{x} - w$
  - the effect of prices and wage changes on utility is only via the budget constraint.
- The Envelope Theorem: $D_{\mathbf{q}}\phi(x^*(\overline{\mathbf{q}});\overline{\mathbf{q}}) = D_{\mathbf{q}}\phi(\mathbf{x},\mathbf{q})\,[-\lambda^*(\mathbf{q})]^\top D_{\mathbf{q}}F(\mathbf{x},\mathbf{q})\Big|_{\substack{\mathbf{x}=x^*(\overline{\mathbf{q}})\\ \mathbf{q}=\overline{\mathbf{q}}}}$
- Thus $\frac{\partial v}{\partial p_i} = \frac{\partial u}{\partial p_i} - \lambda\frac{\partial(\mathbf{p}\cdot\mathbf{x}-w)}{\partial p_i}\Big|_{\substack{\lambda=\lambda^*(\mathbf{p},w)\\ \mathbf{x}=x^*(\mathbf{p},w)}} = 0 - \lambda x_i\big|_{\substack{\lambda=\lambda^*(\mathbf{p},w)\\ \mathbf{x}=x^*(\mathbf{p},w)}} = -\lambda^*(\mathbf{p},w)x_i^*(\mathbf{p},w)$
- and $\frac{\partial v}{\partial w} = \frac{\partial u}{\partial w} - \lambda\frac{\partial(\mathbf{p}\cdot\mathbf{x}-w)}{\partial w}\Big|_{\substack{\lambda=\lambda^*(\mathbf{p},w)\\ \mathbf{x}=x^*(\mathbf{p},w)}} = 0 + \lambda\big|_{\substack{\lambda=\lambda^*(\mathbf{p},w)\\ \mathbf{x}=x^*(\mathbf{p},w)}} = \lambda^*(\mathbf{p},w)$
  the Lagrange multiplier equals the marginal utility of a change in income.
- Divide one of the two expressions above by the other and obtain
  $-x_i^*(\mathbf{p},w) = \frac{-\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial w}}$.
- This is expression gives $\boxed{\text{Roy's Identity: } \frac{\partial v}{\partial p_i} = -x_i^*(\mathbf{p},w)\frac{\partial v}{\partial w}.}$

Hicksian Demand

Definition

Given a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the Hicksian demand correspondence $h^* : \mathbb{R}^n_{++} \times u(\mathbb{R}^n_+) \to \mathbb{R}^n_+$ is defined by

$$h^*(\mathbf{p}, v) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \text{ subject to } u(\mathbf{x}) \geq v.$$

- This finds the cheapest consumption bundle that achieves a given utility level.
- Hicksian demand is also called compensated demand: along it one can measure the impact of price changes for fixed utility.
  - Walrasian demand $x^*(\mathbf{p}, w)$ is also called *uncompensated demand*: along it price changes can make the consumer better-off or worse-off.
- The constraint is in "utils" while the objective function is in money.
  - For Walrasian (uncompensated) demand, the constraint is in money while the objective is in "utils".
- This is the dual of the utility maximization problem:
  - the solutions to the two problems are connected when constraints match.

Hicksian Demand

Definition

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$$h^*(\mathbf{p}, v) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} p \cdot \mathbf{x} \text{ subject to } u(\mathbf{x}) \geq v.$$

Properties of Hicksian Demand

Proposition

If u is continuous, then $h^*(\mathbf{p}, v)$ is nonempty and compact.

Proof.

By continuity, $\{\mathbf{x} \in \mathbb{R}^n_+ : u(\mathbf{x}) \geq v\}$, the upper contour set of $\mathbf{x}$, is closed.

- For a sufficiently large M, the closed set $\{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq M\}$ and the upper contour set of $\mathbf{x}$ are not disjoint.
- Then

$$h^*(\mathbf{p}, v) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \text{ subject to } \begin{array}{c} u(\mathbf{x}) \geq v \\ \text{and} \\ \mathbf{p} \cdot \mathbf{x} \leq M \end{array}$$

- The modified constraint set is closed and bounded.
- ... from here on, the proof follows the proof that Walrasian demand is nonempty and compact... (fill in the details as exercise).

□

Hicksian Demand Is Downward Sloping (by Revealed Preferences)

Law of Demand: if the price of a good increases the compensated demand for that good cannot increase

- Take two price vectors $\mathbf{p}$ and $\mathbf{q}$, and define: $\mathbf{x} \in h^*(\mathbf{p}, v)$ and $\mathbf{y} \in h^*(\mathbf{q}, v)$

1. $\mathbf{y}$ *could have been chosen at prices* $\mathbf{p}$ *but was not, hence* $\mathbf{y}$ *cannot be cheaper than* $\mathbf{x}$ *at prices* $\mathbf{p}$. $\Rightarrow$ $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{y}$
2. $\mathbf{p} \cdot (\mathbf{x} - \mathbf{y}) \leq 0$
3. $\mathbf{x}$ *could have been chosen at prices* $\mathbf{q}$ *but was not, hence* $\mathbf{x}$ *cannot be cheaper than* $\mathbf{y}$ *at prices* $\mathbf{q}$ $\Rightarrow$ $\mathbf{q} \cdot \mathbf{x} \geq \mathbf{q} \cdot \mathbf{y}$
4. $\mathbf{q} \cdot (\mathbf{x} - \mathbf{y}) \geq 0$ *or* $-\mathbf{q} \cdot (\mathbf{x} - \mathbf{y}) \leq 0$
5. *Adding 2. and 4. we get* $(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{x} - \mathbf{y}) \leq 0$
6. *Choose* $\mathbf{p}$ *and* $\mathbf{q}$ *so that* $p_i \neq q_i$ *and* $p_j = q_j$ *for all* $j \neq i$*:*

$$(p_i - q_i)(x_i - y_i) \leq 0$$

- A similar result does not hold for Walrasian demand.

The Expenditure Function

Definition

Given a continuous utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$, the expenditure function $e : \mathbb{R}^n_{++} \times u(\mathbb{R}^n_+) \to \mathbb{R}_+$ is defined by

$$e(\mathbf{p}, v) = \mathbf{p} \cdot \mathbf{x}^*$$

for some $\mathbf{x}^* \in h^*(\mathbf{p}, v)$.

- This function tracks the minimized value of the amount spent by the consumer as prices and utility change.

Proposition

If the utility function is continuous and locally nonsatiated, then the expenditure functions is homogeneous of degree 1 *and concave in* $\mathbf{p}$*.*

Proof.

Question 3 in Problem Set 3. □

Walrasian and Hicksian Demand Are Equal

Proposition

Suppose u is continuous and locally nonsatiated. If $v > u(\mathbf{0}_n)$, then:

$$\begin{aligned} x^*(\mathbf{p}, w) &= h^*(\mathbf{p}, v(\mathbf{p}, w)); \\ h^*(\mathbf{p}, v) &= x^*(\mathbf{p}, e(\mathbf{p}, v)). \end{aligned}$$

Walrasian and Hicksian Demand Are Equal

Proposition

Suppose u is continuous and locally nonsatiated. If $v > u(\mathbf{0}_n)$, then:

$$\begin{aligned} x^*(\mathbf{p}, w) &= h^*(\mathbf{p}, v(\mathbf{p}, w)); \\ h^*(\mathbf{p}, v) &= x^*(\mathbf{p}, e(\mathbf{p}, v)). \end{aligned}$$

- These are sets: the consumption bundles that maximize utility are the same as the consumption bundles that minimize expenditure, provided the constraints of the two problems "match up".
  - The income in the utility maximization problem must be $\mathbf{p} \cdot \mathbf{x}^*$, where $\mathbf{x}^* \in h^*(\mathbf{p}, v)$.
  - The utility in the expenditure minimization problem must be $u(\mathbf{x}^*)$, where $\mathbf{x}^* \in x^*(\mathbf{p}, v)$.
- Walrasian and Hicksian demand coincide if computed according to the same prices, income, and utility.
- The proposition implies that

$$e(\mathbf{p}, v(\mathbf{p}, w)) = w \text{ and } v(\mathbf{p}, e(\mathbf{p}, v)) = v$$

  so for a fixed price vector $\mathbf{p}$, the functions $e(\mathbf{p}, \cdot)$ and $v(\mathbf{p}, \cdot)$ are inverses of each other.

Walrasian and Hicksian Demand Are Equal

We want to show that $\mathbf{x}^* \in x^*(\mathbf{p}, w)$ solves $\boxed{\min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{p}\cdot\mathbf{x} \text{ subject to } u(\mathbf{x}) \geq v}$

Proof.

Pick $\mathbf{x}^* \in x^*(\mathbf{p}, w)$, and suppose $\mathbf{x}^* \notin h^*(\mathbf{p}, u(\mathbf{x}^*))$: $\mathbf{x}^*$ is a utility maximizer but not expenditure minimizer.

- Then $\exists \mathbf{x}'$ s.t.
$$\mathbf{p}\cdot\mathbf{x}' < \mathbf{p}\cdot\mathbf{x}^* \leq w \text{ and } u(\mathbf{x}') \geq u(\mathbf{x}^*)$$
- By local nonsatiation, $\exists \mathbf{x}''$ (close to $\mathbf{x}'$) s.t.
$$\mathbf{p}\cdot\mathbf{x}'' \leq w \text{ and } u(\mathbf{x}'') > u(\mathbf{x}')$$
- contradicting the fact that $\mathbf{x}^*$ maximizes utility. Thus $\mathbf{x}^*$ must also minimize expenditure.
- Finally, since $\mathbf{p}\cdot\mathbf{x}^* = w$ by full expenditure, we also have
$$e(\mathbf{p}, v(\mathbf{p}, w)) = \mathbf{p}\cdot\mathbf{x}^* = w$$

□

Walrasian and Hicksian Demand Are Equal

We want to show that $\mathbf{x}^* \in h^*(\mathbf{p}, v)$ solves $\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x})$ subject to $\mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{x}^*$

Proof.

Pick $\mathbf{x}^* \in h^*(\mathbf{p}, v)$, and suppose $\mathbf{x}^* \notin x^*(\mathbf{p}, \mathbf{p} \cdot \mathbf{x}^*)$: $\mathbf{x}^*$ is an expenditure minimizer but not a utility maximizer.

- Then $\exists \mathbf{x}'$ s.t.

$$u(\mathbf{x}') > u(\mathbf{x}^*) \quad \text{and } \mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \mathbf{x}^*$$

- Consider the bundle $\alpha \mathbf{x}'$ with $\alpha < 1$ but very close to 1. By continuity of u,

$$u(\alpha \mathbf{x}') > u(\mathbf{x}^*) \geq v$$

- Therefore,

$$\mathbf{p} \cdot \alpha \mathbf{x}' < \mathbf{p} \cdot \mathbf{x}' \leq \mathbf{p} \cdot \mathbf{x}^*$$

  contradicting the fact that $\mathbf{x}^*$ minimizes expenditure. Thus $\mathbf{x}^*$ must also maximize utility.

- Finally, because $u(\mathbf{x}^*) = v$ and $\mathbf{p} \cdot \mathbf{x}^* = e(\mathbf{p}, v)$, we have

$$v(\mathbf{p}, e(\mathbf{p}, v)) = v$$

□

Support Function

Definition

Given a closed set $K \subseteq \mathbb{R}^n$, the support function $\mu_K : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is

$$\mu_K(\mathbf{p}) = \inf_{\mathbf{x} \in K} \mathbf{p} \cdot \mathbf{x}.$$

- It can equal $-\infty$ since there might exists $\mathbf{x} \in K$ such that $\mathbf{p} \cdot \mathbf{x}$ becomes unboundedly negative for a closed set.
  - for example: $K = \{\mathbf{x} \in \mathbb{R}^2 : x_1 \in \mathbb{R}, x_2 \in [0, \infty)\}$, and $\mathbf{p} = (-1, 0)$.
- If K is convex (closed and bounded in $\mathbb{R}^n$), the support function is finite.
- When a set K is convex, one can 'recover' it using the support function:
  - given a $\mathbf{p} \in \mathbb{R}^n$, $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{x} \geq \mu_K(p)\}$ is an half space that contains K; furthermore, K is the intersection of all such half spaces (for all p).
  - If K is not convex, the intersection of all sets $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{p} \cdot \mathbf{x} \geq \mu_K(\mathbf{p})\}$ is the smallest closed convex set containing K (this is called the convex hull).

Duality Theorem

Definition

Given a closed set $K \subseteq \mathbb{R}^n$, the support function $\mu_K : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is

$$\mu_K(\mathbf{p}) = \inf_{\mathbf{x} \in K} \mathbf{p} \cdot \mathbf{x}.$$

Theorem (Duality Theorem)

Let K be a nonempty closed set. There exists a unique $\mathbf{x} \in K$ such that $\mathbf{p} \cdot \mathbf{x} = \mu_K(\mathbf{p})$ if and only if μ_K is differentiable at $\mathbf{p}$. If so,

$$\nabla \mu_K(\mathbf{p}) = \mathbf{x}$$

- The support function is ‘linear’ in **p**.

Example

The expenditure function $e(\mathbf{p}, v)$ is the support function of the “better-than" set

$$K = \{\mathbf{x} \in \mathbb{R}^n_+ : u(\mathbf{x}) \geq v\}$$

Properties of the Expenditure Function

Proposition

If $u(\mathbf{x})$ is continuous, locally nonsatiated, and strictly quasiconcave, then $e(\mathbf{p}, v)$ is differentiable in $\mathbf{p}$.

Proof.

Immediate from the previous theorem (verify the assumptions hold). □

Shephard's Lemma

Proposition (Shephard's Lemma)

Suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ is a continuous, locally nonsatiated, and strictly quasiconcave utility function. Then, for all $\mathbf{p} \in \mathbb{R}^n_{++}$ and $v \in \mathbb{R}$,

$$h^*(\mathbf{p}, v) = \nabla_{\mathbf{p}} e(\mathbf{p}, v).$$

- Hicksian demand is the derivative of the expenditure function.

There are different ways to prove Shephard's Lemma:

- Use the duality theorem.
- Use the envelope theorem:
  - let $\phi(\mathbf{x}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{x}$, $\phi(x^*(\mathbf{q}); \mathbf{q}) = e(\mathbf{p}, v)$, and $F(\mathbf{x}, \mathbf{q}) = u(\mathbf{x}) - v$,
  - then:

    $$D_{\mathbf{q}}\phi(x^*(\bar{\mathbf{q}}); \bar{\mathbf{q}}) = D_{\mathbf{q}}\phi(\mathbf{x}, \mathbf{q})\big|_{\mathbf{x}=x^*(q), \mathbf{q}=\bar{\mathbf{q}}} - [\lambda^*(\bar{\mathbf{q}})]^\top D_{\mathbf{q}}F(\mathbf{x}, \mathbf{q})\big|_{\mathbf{x}=x^*(\bar{q}), \mathbf{q}=\bar{\mathbf{q}}}$$

  - becomes

    $$\nabla_{\mathbf{p}} e(\mathbf{p}, v) = h^*(\mathbf{p}, v) - 0$$

    since F does not depend on $\mathbf{p}$.
- Brute force (next slide).

Shephard's Lemma: Proof

Proof.

We want to show that $h^*(\mathbf{p}, v) = \nabla_{\mathbf{p}} e(\mathbf{p}, v)$.

- Using the definition, and then the chain rule
$$\nabla_{\mathbf{p}} e(\mathbf{p}, v) = \nabla_{\mathbf{p}} \left[\mathbf{p} \cdot h^* (\mathbf{p}, u)\right] = h^* (\mathbf{p}, u) + \left[\mathbf{p} \cdot D_{\mathbf{p}} h^* (\mathbf{p}, u)\right]^\top$$
- The first order conditions of the minimization problem say
$$\mathbf{p} = \lambda \nabla_{\mathbf{p}} u \left(h^* (\mathbf{p}, u)\right)$$
- Therefore
$$\nabla_{\mathbf{p}} e(\mathbf{p}, v) = h^* (\mathbf{p}, u) + \lambda \left[\nabla_{\mathbf{p}} u \left(h^* (\mathbf{p}, u)\right) \cdot D_{\mathbf{p}} h^* (\mathbf{p}, u)\right]^\top$$
- At an optimum, the constraint must bind and so
$$u \left(h^* (\mathbf{p}, u)\right) = v$$
- Thus:
$$\nabla_{\mathbf{p}} u \left(h^* (\mathbf{p}, u)\right) \cdot D_{\mathbf{p}} h^* (\mathbf{p}, u) = 0$$
- and therefore:
$$h^*(\mathbf{p}, v) = \nabla_{\mathbf{p}} e(\mathbf{p}, v).$$

  as desired. □

Proposition (Roy's identity)

Suppose u is continuous, locally nonsatiated, and strictly quasiconcave and v is differentiable at $(\mathbf{p}, w) \neq 0$. Then $\boxed{x^*(\mathbf{p}, w) = -\nabla_{\mathbf{p}} v(\mathbf{p}, v) \frac{1}{\frac{\partial v(\mathbf{p}, w)}{\partial w}}}$

- This can be also written as
$$x_k^*(\mathbf{p}, w) = -\frac{\partial v(\mathbf{p}, w)}{\partial p_k} \frac{1}{\frac{\partial v(\mathbf{p}, w)}{\partial w}}, \qquad \text{for all } k$$
- Walrasian demand equals the derivative of the indirect utility function multiplied by a "correction term".
  - This correction normalizes by the marginal utility of wealth.

There are different ways to prove Roy's Identity

- Use the envelope theorem (earlier).
- Use the chain rule and the first order conditions.
- Brute force (next slide).

Roy's Identity

Proof.

We want to show that $x_k^*(\mathbf{p}, w) = -\frac{\frac{\partial v(\mathbf{p},w)}{\partial p_k}}{\frac{\partial v(\mathbf{p},w)}{\partial w}}$. Fix some $\bar{\mathbf{p}}, \bar{w}$ and let $\bar{u} = v(\bar{\mathbf{p}}, \bar{w})$.

- The following identity holds for all $\mathbf{p}$
$$v(\mathbf{p}, e(\mathbf{p}, \bar{u})) = \bar{u}$$
- differentiating w.r.t p_k we get
$$\frac{\partial v}{\partial p_k} + \frac{\partial v}{\partial w}\frac{\partial e}{\partial p_k} = 0$$
- and therefore
$$\frac{\partial v}{\partial p_k} + \frac{\partial v}{\partial w}h_k = 0$$
- Let $\bar{\mathbf{x}}_k = x_k(\bar{\mathbf{p}}, \bar{w})$ and evaluate the previous equality at $\bar{\mathbf{p}}, \bar{w}$:
$$\frac{\partial v}{\partial p_k} + \frac{\partial v}{\partial w}\bar{x}_k = 0$$
- Solve for $\bar{x}_k$ to get the result. □

Slutsky Matrix

Definition

The Slutsky matrix, denoted $D_{\mathbf{p}} h^*(\mathbf{p}, v)$, is the $n \times n$ matrix of derivative of the Hicksian demand function with respect to price (its first n dimensions).

- Notice this says "function", so the Slutsky matrix is defined only when Hicksian demand is unique.

Proposition

Suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous, locally nonsatiated, strictly quasiconcave and $h^*(\cdot)$ is continuously differentiable at $(\mathbf{p}, v)$. Then:

1. *The Slutsky matrix is the Hessian of the expenditure function:* $D_{\mathbf{p}} h^*(\mathbf{p}, v) = D^2_{\mathbf{pp}} e(\mathbf{p}, v)$;
2. *The Slutsky matrix is symmetric and negative semidefinite;*
3. $D_{\mathbf{p}} h^*(\mathbf{p}, v)\mathbf{p} = \mathbf{0}_n$.

Proof.

Question 5, Problem Set 3. □

Slutsky Decomposition

Proposition

Assume u is continuous, locally nonsatiated, and strictly quasiconcave, and that h^* is differentiable. Then, for all $(\mathbf{p}, w)$

$$D_{\mathbf{p}} h^*(\mathbf{p}, v(\mathbf{p}, w)) = D_{\mathbf{p}} x^*(\mathbf{p}, w) + D_w x^*(\mathbf{p}, w) x^*(\mathbf{p}, w)^\top$$

or

$$\frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_k} = \frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_k} + \frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_k^*(\mathbf{p}, w), \qquad \text{for all } j, k$$

Proof.

Remember that $h^*(\mathbf{p}, v) = x^*(\mathbf{p}, e(\mathbf{p}, v))$.

- Take the equality for good j and differentiate with respect to p_k
$$\frac{\partial h_j^*}{\partial p_k} = \frac{\partial x_j^*}{\partial p_k} + \frac{\partial x_j^*}{\partial w}\frac{\partial e}{\partial p_k} = \frac{\partial x_j^*}{\partial p_k} + \frac{\partial x_j^*}{\partial w} h_k^*$$
- Evaluate this at $\mathbf{p}, w$ and $u = v(\mathbf{p}, w)$ so that $h^*(\cdot) = x^*(\cdot)$
$$\frac{\partial h_j^*}{\partial p_k} = \frac{\partial x_j^*}{\partial p_k} + \frac{\partial x_j^*}{\partial w} x_k^*$$

□

Slutsky Equation

Hicksian decomposition of demand

Rearranging from the previous proposition:

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_k} = \underbrace{\frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_k}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_k^*(\mathbf{p}, w)}_{\text{income effect}}$$

- This is also known as the Slutsky equation:
  - it connects the derivatives of compensated and uncompensated demands.
- If one takes $k = j$, the following is the "own price" Slutsky equation

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_j} = \underbrace{\frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_j}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_j^*(\mathbf{p}, w)}_{\text{income effect}}$$

Normal, Inferior, and Giffen Goods

The own price Slutsky equation

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_j} = \underbrace{\frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_j}}_{\text{substitution effect}} - \underbrace{\frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_j^*(\mathbf{p}, w)}_{\text{income effect}}.$$

- Since the Slutsky matrix is negative semidefinite, $\partial h_k^* / \partial p_k \leq 0$;
  - the first term is always negative while the second can have either sign;
  - the substitution effect always pushes the consumer to purchase less of a commodity when its price increases.

Normal, Inferior, and Giffen Goods

- A normal good has a positive income affect.
- An inferior good has a negative income effect.
- A Giffen good has a negative overall effect (i.e. $\partial x_k^* / \partial p_k > 0$);
  - this can happen only if the income effect is negative and overwhelms the substitution effect

$$\frac{\partial x_k^*}{\partial w} x_k^* < \frac{\partial h_k^*}{\partial p_k} \leq 0.$$

Slutsky Equation and Elasticities

Definition

$\varepsilon_{y,q} = \frac{\partial y}{\partial q}\frac{q}{y}$ is called the elasticity of y with respect to q

- Elasticity is a unit free measure (percentage change in y for a given percentage change in q) that is often used to compare price effects across different goods.

The own price Slutsky equation

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_j} = \frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_j} - \frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_j^*(\mathbf{p}, w)$$

- Multiply both sides by $p_j / x_j^*(\cdot)$ and rewrite as

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_j}\frac{p_j}{x_j^*(p, w)} = \frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_j}\frac{p_j}{x_j^*(\mathbf{p}, w)} - \frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} p_j \frac{w}{x_j^*(\mathbf{p}, w)}\frac{x_j^*(\mathbf{p}, w)}{w}$$

or

$$\varepsilon_{x_j,p_j} = \varepsilon_{h_j,p_j} - \varepsilon_{x_j,w}\frac{p_j x_j^*(\mathbf{p}, w)}{w}$$

  - The elasticty of uncompensated demand equals the elasticity of compensated demand minus the income elasticity of Walrasian demand multiplied by that good share in the budget.

Gross Substitutes

The cross price Slutsky equation

$$\frac{\partial x_j^*(\mathbf{p}, w)}{\partial p_k} = \frac{\partial h_j^*(\mathbf{p}, v(\mathbf{p}, w))}{\partial p_k} - \frac{\partial x_j^*(\mathbf{p}, w)}{\partial w} x_j^*(\mathbf{p}, w)$$

Intuitively, we think of two goods as substitutes if the demand for one increases when the price of the other increases.

Definition

We say good j is a gross substitute for k if $\frac{\partial x_j^*(\mathbf{p},w)}{\partial p_k} \geq 0$.

- Unfortunately the definition of substitutes based on Walrasian (uncompensated) demand is not very useful since it does not satisfy symmetry: we can have $\frac{\partial x_j^*(\mathbf{p},w)}{\partial p_k} > 0$ and $\frac{\partial x_k^*(\mathbf{p},w)}{\partial p_j} < 0$.
- There is a better definition that is based on Hicksian (compensated) demand.

Net Substitutes

Definition

We say goods j and k are

net substitutes if $\frac{\partial h_j^*(\mathbf{p},w)}{\partial p_k} \geq 0$ and net complements if $\frac{\partial h_j^*(\mathbf{p},w)}{\partial p_k} \leq 0$.

This definition is symmetric: $\frac{\partial h_j^*(\mathbf{p},w)}{\partial p_k} \geq 0$ *if and only if* $\frac{\partial h_k^*(\mathbf{p},w)}{\partial p_j} \geq 0$.

Proof.

$$\begin{aligned}
\frac{\partial h_j^*(\mathbf{p},w)}{\partial p_k} &= \frac{\partial \frac{\partial e(\mathbf{p},v)}{\partial p_j}}{\partial p_k} && \text{by Shephard's Lemma} \\
&= \frac{\partial^2 e(\mathbf{p},v)}{\partial p_k \partial p_j} \\
&= \frac{\partial^2 e(\mathbf{p},v)}{\partial p_j \partial p_k} && \text{by Young's Theorem} \\
&= \frac{\partial h_k^*(\mathbf{p},w)}{\partial p_j} && \text{by Shephard's Lemma again}
\end{aligned}$$

□

Next Class

- Comparative Statics Without Calculus
- Testable Implication of Consumer Theory